UK Junior Mathematical Olympiad 2006 Solutions

A1 **2006** $1 + 2 \times (3 + 4^5) + 6 + 7 - 8 \times 9 + 10 = 1 + 2 \times 1027 + 6 + 7 - 72 + 10$ $= 1 + 2054 - 49 = 2006$.

A2 **27cm³** Let the length of each edge of the cube be x cm. Then $14x = 42$, so $x = 3$.

A3 **792** The combinations of three digits which have a product of 36 are (1, 4, 9), (1, 6, 6), (2, 2, 9), (2, 3, 6), (3, 3, 4). So the greatest and smallest three-digit numbers for which the product of the digits equals 36 are 941 and 149 respectively.

A4 **11** If all three regions are painted the same colour, then three different painted triangles are possible. If exactly two regions are the same colour, there are three choices for that colour and, for each choice, the remaining region can be coloured two ways, giving six possibilities in all. If all three regions are different colours, there are only two possible painted triangles. Going clockwise round the triangle one will be red, yellow then blue and the other will be red, blue and then yellow. Any other painted triangle with all three colours used will be a rotation of one of these and so not counted as different. Hence a total of 11 different painted triangles may be made.

A5 **120** The balloon seller sells 1/3 of his balloons to boys and 1/5 of them to girls. The difference between these two fractions is 2/15, so he sells 6/15, that is 2/5, of the balloons to adults. The total number of balloons sold, therefore, is 14/15 of the number he started with. This means that the 8 balloons which remain represent 1/15 of the original number of balloons.

A6 **10°** The star is a 12-sided polygon, so the sum of its interior angles $= (12 - 2) \times 180^\circ = 1800^\circ$. Each of its reflex angles is $360^\circ - 70^\circ$, that is 290°. So each of the angles marked '$\circ$' $= (1800^\circ - 6 \times 290^\circ) \div 6 = 10^\circ$.

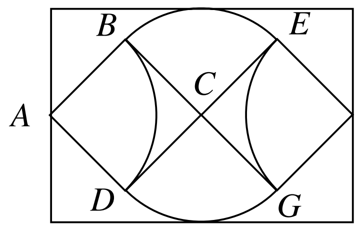
A7 **£7** The only factors of 781 are 1, 11, 71 and 781 so we can conclude that Tom is 11 years old and his grandfather is 71 years old. Last year they were 10 and 70 years old respectively, so Tom received 700p.

A8 **4** The maximum sum for any row or column is $7 + 8 + 9 = 24$, so these three numbers in some order make up row 3. Given that, the maximum sum for any column is $9 + 6 + 5 = 20$, so those three numbers in some order make up column three. Hence $i = 9$. The largest total for any other row is $6 + 4 + 3 = 13$, so those three numbers in some order make up row 2. Hence $f = 6$. The largest total for another column is $8 + 4 + 2 = 14$, so those three numbers in some order make up column 2. Hence $e = 4$.

a	b	c	8
d	e	f	13
g	h	i	24
11	14	20	

A9 **23** Three different prime numbers, one of which is 2, have a sum which is even, so we may conclude that 2 does not appear in either set of three prime numbers. The next prime numbers in ascending order are 3, 5, 7, 11 so the required number must be at least $3 + 5 + 11$, that is 19. However, this is the only way of writing 19 as the sum of three different primes. The smallest prime greater than 19 is 23, which equals $3 + 7 + 13$ and also equals $5 + 7 + 11$. So 23 is the smallest prime which may be written as the sum of three different primes in two different ways.

A10 $\sqrt{2}$ **m** The radius of each of the quarter circles is 50cm. So by symmetry we see that $ABCD$ and $CEFG$ are both squares of side 50cm.



From Pythagoras' Theorem:

$AC^2 = (50^2 + 50^2)\text{ cm}^2 = 5000\text{ cm}^2$.

So $AC = \sqrt{5000}\text{ cm} = 50\sqrt{2}\text{ cm}$.

Similarly, $CF = 50\sqrt{2}$ cm, so

$AF = 100\sqrt{2}\text{ cm} = \sqrt{2}\text{ m}$.

B1 Let the number of 10p and 20p coins which Jenny has be x and y respectively. Then the number of 50p coins she has is $(20 - x - y)$.

So $10x + 20y + 50(20 - x - y) = 500$, that is $x + 2y + 5(20 - x - y) = 50$.

So x and y are positive integers which satisfy the equation $4x + 3y = 50$.

The possibilities are $x = 11, y = 2$; $x = 8, y = 6$; $x = 5, y = 10$; $x = 2, y = 14$.

The number of 50p coins corresponding to these are 7, 6, 5, 4 respectively and as Jenny has more 50p coins than 10p coins, we may conclude that she has two 10p coins, fourteen 20p coins and four 50p coins.

B2 For the final number in the chain to be 6, the penultimate number must be 16, 23, 32 or 61.

Any of these could be the first number in a chain of length 2, but 16 or 32 could be the penultimate number in a longer chain. However, 23 and 61 are prime, so cannot be the product of the digits of a previous number.

If the penultimate number is 16, the previous number must be 28, 44 or 82. Of these, only 28 is the product of two single digits and would be the next number in a chain after 47 or 74. In either case, 28 would be the second number in the chain since 47 is prime and 74 cannot be written as the product of two single digits.

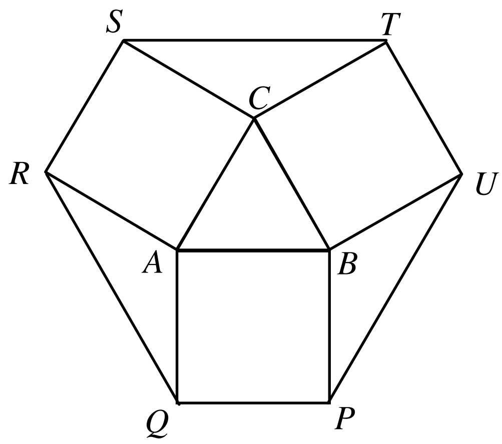
If the penultimate number is 32, the previous number must be 48 or 84. Of these, only 48 is the product of two single digits and would be the next number in a chain after 68 or 86. In either case, 32 would be the second number in the chain since neither 68 nor 86 may be written as the product of two single digits.

So the possible two-digit first numbers for a chain are, in ascending order: 16, 23, 28, 32, 44, 47, 48, 61, 68, 74, 82, 84, 86.

B3 Let $\angle XBC = x°$. Then $\angle BXY = 180° - x°$ as these two angles are angles inside parallel lines XY and BC.

Also, $\angle AXY = 7x°$ (given), so $\angle AXB = [360° - (180° - x°) - 7x°] = 180° - 6x°$ as angles at a point total 360°. Hence in isosceles triangle ABX, angles XBA and XAB total $[180° - (180° - 6x°)]$, that is $6x°$. So, since $AX = BX$, $\angle XBA = \angle XAB = 3x°$. Furthermore, as AX bisects angle BAC, $\angle XAY = 3x°$. So in triangle AXY, $7x + 3x + 90 = 180$; hence $x = 9$. The size of angle ABC, therefore, is $4 \times 9°$, that is 36°.

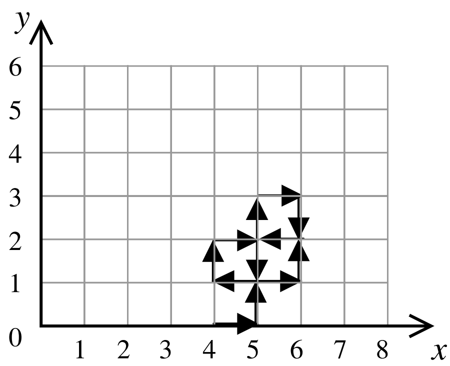
B4 Consider triangle STC: the angles meeting at C total 360°, so $\angle SCT = (360 - 60 - 90 - 90)°$, that is 120°. As $CS = CT = 2$, the perpendicular from C to TS divides triangle STC into two congruent triangles, each having interior angles of 30°, 60°, 90°. So these two triangles may be placed together to form an equilateral triangle of side 2; therefore triangle STC is equal in area to triangle ABC, as, by symmetry, are triangles AQR and BPU.



By Pythagoras' Theorem, the perpendicular height of triangle $ABC = \sqrt{2^2 - 1^2} = \sqrt{3}$, so the area of triangle ABC is $\frac{1}{2} \times 2 \times \sqrt{3}$, that is $\sqrt{3}$. Each of the squares $ABPQ$, $BCTU$, $CARS$ has area 4, so the area of hexagon $PQRSTU$ is $12 + 4\sqrt{3}$.

B5 (a) The diagram shows the points visited in the first 12 moves.

After (4, 0), the points visited, in order, are (5, 0), (5, 1), (4, 1), (4, 2), (5, 2), (5, 1), (6, 1), (6, 2), (5, 2), (5, 3), (6, 3), (6, 2). So it is at (6, 2) after 12 moves and is facing 'East'.

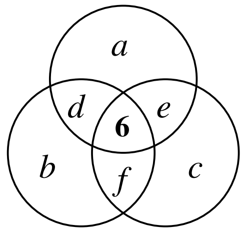


(b) Note that the effect of the first 6 moves is to move the bug 1 unit in both the positive x and y directions and to leave it facing 'East'. As the value of x has increased by 1 and the value of y has also increased by 1, the series of values which $(x - y)$ takes over the next 6 moves will be the same as those for the first 6 moves with the effect that the bug will once again move 1 unit in both the positive x and y directions and will face 'East' after this next series of 6 moves. This process will then continue for the next 6 moves and so on.

So, as has already been seen, the bug will be at the point (6, 2) after 12 moves. It will subsequently be at (7, 3) after 18 moves, at (8, 4) after 24 moves and so on. So it will be at (12, 8) after 48 moves.

As it is now facing 'East' it will be at (13, 8) after the 49th move and will then turn left and move to (13, 9), its position after 50 moves.

B6 Let the numbers inside the regions be a, b, c, d, e, f as shown.
Then: $a + d + e + 6 = T$; $b + d + f + 6 = T$; $c + e + f + 6 = T$.



Adding these equations gives $a + b + c + 2d + 2e + 2f + 18 = 3T$.
Now a, b, c, d, e, f are 1, 2, 3, 4, 5, 7 in some order, so $a + b + c + d + e + f = 22$.

Therefore $22 + d + e + f + 18 = 3T$, that is $40 + d + e + f = 3T$.

Now the minimum value of $d + e + f = 1 + 2 + 3 = 6$ and the maximum value $= 4 + 5 + 7 = 16$. So $46 \leqslant 3T \leqslant 56$. Since T is a positive integer, the only values it can have is 16, 17 or 18.

If $T = 16$, then $d + e + f = 8$ and the task may be completed with $a = 7, b = 4, c = 3, d = 1, e = 2, f = 5$.

If $T = 17$, then $d + e + f = 11$. However, $a + d + e + 6 = T = 17$, so $a + d + e = 11$. This requires a to equal f, which is impossible, so T cannot be 17.

If $T = 18$, then $d + e + f = 14$ and the task may be completed with $a = 5, b = 2, c = 1, d = 3, e = 4, f = 7$.

Thus the only possible values of T are 16 and 18.